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Pairwise entanglement in symmetric multi-qubit systems

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Abstract. For pairs of particles extracted from a symmetric state of N two-level systems, the two-particle density matrix is expressed in terms of expectation values of collective spin operators $\mathbf{S}$ for the large system. Results are presented for experimentally relevant examples of pure states: Dicke states $|S, M\rangle$, spin coherent, and spin squeezed states, where only the symmetric subspace, $S = N/2$ is populated, and for thermally entangled mixed states populating also lower S values. The entanglement of the extracted pair is then quantified by a calculation of the concurrence, which provides directly the entanglement of formation of the pair.

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1 Introduction

Various proposals exist for the preparation of multi-particle entangled states, and a number of these states have been pointed out to be particularly easy to prepare and to have special and useful properties [1–11]. Since entanglement is defined as a property of the whole ensemble of particles, it is not immediately clear whether a sub-ensemble of particles, drawn at random from the original ensemble will also be in an entangled state, or whether the trace over some particles will destroy the quantum correlations in the system. Substantial efforts have been made to quantify multi-particle entanglement and to explore how multi-particle entanglement manifests itself under different partitionings of the system, and to which extent different kinds of entanglement may be converted into each other [12–15]. In this paper we consider the simpler question whether a random pair of particles, extracted from a symmetric state of N two-level systems will be in an entangled state or not, where by symmetric, we assume symmetry under any permutation of the particles. We present a method to express the two-particle density matrix in terms of expectation values of collective variables of the large ensemble, and we determine the concurrence [16,17] of the pair for a number of physical examples. Entangled states constitute a valuable resource in quantum information processing [18], and the transfer of entanglement between few qubits and the quasi-continuous variables by which we describe many-particle systems, may become an important ingredient in, *e.g.*, quantum data-storage and inter-species teleportation.

The paper is organized as follows. In Section 2, we present the concurrence, introduced by Wootters [16,17], who demonstrated its one-to-one correspondence with the entanglement of formation of a pair of qubits. In Section 3, we show how the density matrix of a pair of qubits can be expressed in terms of expectation values of collective spin operators on the multi-qubit state. In Section 4, we analyze three examples of pure states of the N particles: spin coherent states, Dicke states, and spin squeezed states. In Section 5, we consider an example of a mixed state with thermal entanglement [19–23], and we show examples where the pairwise entanglement depends on the temperature of the system. Finally in Section 6, we assume two separate ensembles in an Einstein-Podolsky-Rosen state of correlated angular momentum components, and we show that a single pair with an atom from each ensemble will be in an entangled state.

2 Two-particle density matrices and entanglement

It is easy to check if a pure state of two quantum systems is an entangled state or not, by simply observing the eigenvalues r_i of the reduced density matrix of either system. It is also possible to quantity the amount or degree of entanglement of the state [24], $E = -\sum_i r_i \log_2 r_i$, which presents the asymptotic ratio between n and m, where n is the number of pairs in the desired state, synthesized from m pairs of maximally entangled states.

For a mixed state with density matrix ρ_{12}, a similar measure can be defined as the minimum value of the weighted average of E over wave functions by which the two-particle density matrix can be written as a weighted sum. It is necessary to search for the minimum, since ρ_{12} can be written in many ways as a weighted sum of pure

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state projections. In the general case, this is a highly non-trivial task, as is the determination whether the state is entangled at all. For two qubits, however, entanglement is equivalent with the non-positivity of the partially transposed density matrix [25], and the entanglement of formation can be obtained as a simple analytical expression [16,17]

$$E = h\left(\frac{1+\sqrt{1-\mathcal{C}^2}}{2}\right) \tag{1}$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$, and where the *concurrence*, $\mathcal{C}$, is defined as

$$\mathcal{C} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2}$$

where the quantities λ_i are the square roots of the eigenvalues of the matrix product

$$\varrho_{12} = \rho_{12}(\sigma_{1y}\otimes\sigma_{2y})\rho_{12}^*(\sigma_{1y}\otimes\sigma_{2y}) \tag{3}$$

in descending order. In (3) ρ_{12}^* denotes complex conjugation of ρ_{12}, and σ_{iy} are Pauli matrices for the two-level systems. The eigenvalues of ϱ_{12} are real and non-negative even though ϱ_{12} is not necessarily Hermitian, and the values of the concurrence range from zero for an unentangled state to unity for a maximally entangled state.

3 Density matrix for a pair of qubits from a multi-qubit state

A state of two qubits which is symmetric under exchange of the systems has the density matrix

$$\rho_{12} = \begin{pmatrix} v_+ & x_+^* & x_+^* & u^* \\ x_+ & w & y & x_-^* \\ x_+ & y & w & x_-^* \\ u & x_- & x_- & v_- \end{pmatrix} \tag{4}$$

where the matrix elements in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ can be represented by expectation values of Pauli spin matrices of the two systems

$$\begin{aligned} v_\pm &= \frac{1}{4}\left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ x_\pm &= \frac{1}{2}(\langle\sigma_{1+}\rangle \pm \langle\sigma_{1+}\sigma_{2z}\rangle), \\ w &= \frac{1}{4}\left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ y &= \frac{1}{4}\left(\langle\sigma_{1x}\sigma_{2x}\rangle + \langle\sigma_{1y}\sigma_{2y}\rangle\right), \\ u &= \frac{1}{4}(\langle\sigma_{1x}\sigma_{2x}\rangle - \langle\sigma_{1y}\sigma_{2y}\rangle + \mathrm{i}2\langle\sigma_{1x}\sigma_{2y}\rangle)\cdot \end{aligned} \tag{5}$$

We now consider such a state of two qubits which have been extracted from a symmetric multi-qubit states. If only symmetric pure states are considered, we can describe the state of the N-qubit system in terms of the orthonormal basis $|S, M\rangle$ ($M = -S, -S+1, ..., S$) with $S = N/2$. The states $|S, M\rangle$ are the usual symmetric Dicke state [26], *i.e.*, eigenstates of the collective spin operators $\mathbf{S}^2$ and S_z, defined as

$$S_\alpha = \frac{1}{2}\sum_{i=1}^{N}\sigma_{i\alpha}, \quad \alpha = x, y, z. \tag{6}$$

For later use it is convenient to define the number operator $\mathcal{N} = S_z + N/2$ and number states as

$$\begin{aligned} |n\rangle_N &\equiv |N/2, -N/2+n\rangle_N, \\ \mathcal{N}|n\rangle_N &= n|n\rangle_N. \end{aligned} \tag{7}$$

The eigenvalue n of the number operator $\mathcal{N}$ is the number of qubits in the state $|0\rangle$. For example, the states $|0\rangle_N$ and $|1\rangle_N$ are explicitly written as

$$|0\rangle_N = |111...1\rangle, \tag{8}$$

$$\begin{aligned} |1\rangle_N = \frac{1}{\sqrt{N}}(&|011...1\rangle + |101..1\rangle \\ &+ ... + |111...0\rangle). \end{aligned} \tag{9}$$

$|1\rangle_N$ is also called an N-qubit W state [1,20].

Due to the symmetry of the multi-qubit state under exchange of particles we have

$$\begin{aligned} \langle\sigma_{1\alpha}\rangle &= \frac{2\langle S_\alpha\rangle}{N}, \\ \langle\sigma_{1+}\rangle &= \frac{\langle S_+\rangle}{N}, \\ \langle\sigma_{1\alpha}\sigma_{2\alpha}\rangle &= \frac{4\langle S_\alpha^2\rangle - N}{N(N-1)}, \\ \langle\sigma_{1x}\sigma_{2y}\rangle &= \frac{2\langle[S_x, S_y]_+\rangle}{N(N-1)}, \\ \langle\sigma_{1+}\sigma_{2z}\rangle &= \frac{\langle[S_+, S_z]_+\rangle}{N(N-1)}, \end{aligned} \tag{10}$$

where $[A, B]_+ = AB + BA$ is the anticommutator for operators A and B.

From equations (5, 10) we may thus express the density matrix elements of ρ_{12} in terms of the expectation values of the collective operators,

$$\begin{aligned} v_\pm &= \frac{N^2 - 2N + 4\langle S_z^2\rangle \pm 4\langle S_z\rangle(N-1)}{4N(N-1)}, \\ x_\pm &= \frac{(N-1)\langle S_+\rangle \pm \langle[S_+, S_z]_+\rangle}{2N(N-1)}, \\ w = y &= \frac{N^2 - 4\langle S_z^2\rangle}{4N(N-1)}, \\ u &= \frac{\langle S_x^2 - S_y^2\rangle + i\langle[S_x, S_y]_+\rangle}{N(N-1)} = \frac{\langle S_+^2\rangle}{N(N-1)}\cdot \end{aligned} \tag{11}$$

In these expressions we have used the identity $\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{N}{2}(\frac{N}{2}+1)$ valid in the symmetric state space of the particles.

4 Three examples

4.1 Spin coherent states

The spin coherent state [27] is obtained by a rotation of the spin state $|S, M = S\rangle$, which in turn is the product state of all N particles in the $|0\rangle$ state. Hence it is a separable state. It is still interesting to go through the above procedure and to insert the explicit expression of the spin coherent state [27],

$$|\eta\rangle = (1+|\eta|^2)^{-N/2} \sum_{n=0}^{N} \binom{N}{n}^{1/2} \eta^n |n\rangle_N, \qquad (12)$$

where η is chosen real in the following. By a straightforward calculation from equations (11, 12), we find

$$\rho_{12} = \frac{1}{(1+\eta^2)^2} \begin{pmatrix} \eta^4 & \eta^3 & \eta^3 & \eta^2 \\ \eta^3 & \eta^2 & \eta^2 & \eta \\ \eta^3 & \eta^2 & \eta^2 & \eta \\ \eta^2 & \eta & \eta & 1 \end{pmatrix} \qquad (13)$$

which is in agreement with our observation that the two-particle state is really a product state of two rotated spin-$\frac{1}{2}$ particles in the states $(\eta|0\rangle + |1\rangle)/\sqrt{1+\eta^2}$. The matrix product ϱ_{12} is found to be a 4×4 matrix of zero's, revealing the role of the σ_y Pauli matrices in (2): ρ_{12} is the projection operator on spin states with a definite direction in the xz-plane, the application of σ_y is equivalent to a 180° rotation in the xz-plane, and ϱ_{12} is therefore the vanishing product of projection operators on two orthogonal subspaces. Naturally, the concurrence vanishes in this case, $\mathcal{C} = 0$. In other words, there is no pairwise entanglement in the SCS.

4.2 Dicke state $|N/2, M\rangle$

The Dicke states, defined as effective number states above, are states with a definite number of particles occupying the internal states $|0\rangle$ and $|1\rangle$. Such states may in principle be prepared in an atomic physics experiment by Quantum Non-Demolition detection of the atomic populations by phase contrast imaging of the atomic sample [2,3]. By rotation of all spins, a separable spin coherent state is prepared with a binomial distribution on the various Dicke states, *cf.* equation (12), and experiments have already demonstrated a factor 3 reduction in the variance of the populations obtained by such a detection [4].

From equations (4, 11), it is easy to see that the reduced density matrix ρ_{12} is given by

$$\rho_{12} = \begin{pmatrix} v_+ & 0 & 0 & 0 \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ 0 & 0 & 0 & v_- \end{pmatrix} \qquad (14)$$

with matrix elements

$$v_\pm = \frac{(N \pm 2M)(N-2 \pm 2M)}{4N(N-1)},$$
$$y = \frac{N^2 - 4M^2}{4N(N-1)}. \qquad (15)$$

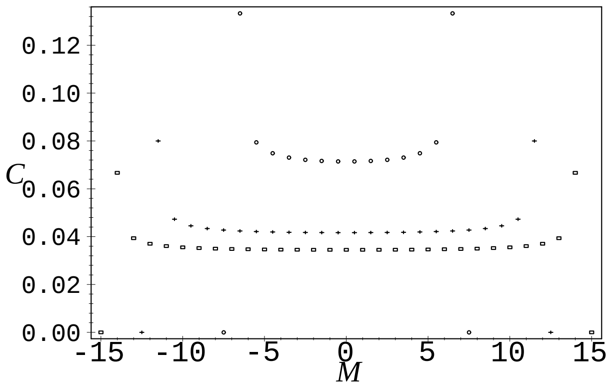

Fig. 1. The concurrence in the Dicke state for different number N. $N = 15$ (open circles), $N = 25$ (crosses), and $N = 30$ (open square).

The corresponding concurrence of a simple density matrix of the form (14) is given by [28]

$$\mathcal{C} = 2\max\{0, y - \sqrt{v_+ v_-}\}, \qquad (16)$$

where we have used the fact $2y + v_+ + v_- = 1$. Now substituting equation (15) into (16), we explicitly obtain

$$\mathcal{C} = \frac{1}{2N(N-1)} \Big\{ N^2 - 4M^2 - \sqrt{(N^2-4M^2)[(N-2)^2 - 4M^2]} \Big\} \cdot \qquad (17)$$

The values of $\mathcal{C}$ for different N and M are illustrated in Figure 1. For any Dicke state except the ones with maximum $|M|$, if one extracts two particles, they will be in an entangled state.

The variation of M around $M = 0$ is small for an initial binomial distribution with this mean value, and the concurrence will be very close to the exact result, $\mathcal{C} = 1/(N-1)$ for $M = 0$, irrespective of the outcome of a QND measurement of M.

The Dicke states $|N/2, M = \pm(N/2-1)\rangle$ have a concurrence of $\mathcal{C} = 2/N$. These states are identical with the W state (see Eq. (9)), which are known to be the symmetric states with the highest possible concurrence [29].

4.3 Kitagawa-Ueda state

In 1993, Kitagawa and Ueda proposed a nonlinear Hamiltonian χS_x^2 in order to generate spin squeezed states [5]. This effective Hamiltonian may be realized in ion traps [6], where it was already implemented in order to produce multi-particle entangled states (of four particles) [7], and it may be implemented in two-component Bose-Einstein condensates as a direct consequence of the collisional interactions between the particles [8], see also [9].

When the Hamiltonian $H = \chi S_x^2$ is applied to the many-particle system, which has been prepared in the product state $|0\rangle_N = |111, ..., 1\rangle$, the wave function at time t is obtained as

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\chi t S_x^2} |0\rangle_N. \qquad (18)$$

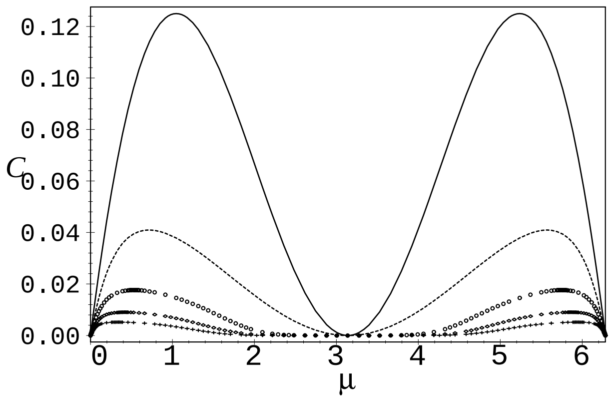

Fig. 2. The concurrence as a function of μ for different number N. $N = 3$ (solid line), $N = 4$ (dashed line), $N = 5$ (open circles), $N = 6$ (open diamonds), and $N = 7$ (crosses).

Using the results obtained in [5] the following expectation values are obtained ($\mu = 2\chi t$)

$$\begin{aligned}
\langle S_x\rangle &= \langle S_y\rangle = 0,\\
\langle S_z\rangle &= -\frac{N}{2}\cos^{N-1}\left(\frac{\mu}{2}\right)\\
\langle S_x^2\rangle &= N/4\\
\langle S_y^2\rangle &= \frac{1}{8}\left(N^2+N-N(N-1)\cos^{N-2}\mu\right)\\
\langle S_z^2\rangle &= \frac{1}{8}\left(N^2+N+N(N-1)\cos^{N-2}\mu\right)\\
\langle [S_+,S_z]_+\rangle &= 0\\
\langle [S_x,S_y]_+\rangle &= \frac{1}{2}N(N-1)\cos^{N-2}\frac{\mu}{2}\sin\frac{\mu}{2}\cdot
\end{aligned} \tag{19}$$

We are now able to determine the two-particle density matrix, which is on the form

$$\rho_{12} = \begin{pmatrix} v_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & v_- \end{pmatrix} \tag{20}$$

with matrix elements given by equation (11). The combination of equations (11, 19) gives explicitly the matrix elements.

From equations (2, 3), the concurrence for the matrix (20) is obtained as

$$\mathcal{C} = \begin{cases} 2\max(0, |u| - y), & \text{if } 2y < \sqrt{v_+v_-} + |u|; \\ 2\max(0, y - \sqrt{v_+v_-}), & \text{if } 2y \geq \sqrt{v_+v_-} + |u|. \end{cases} \tag{21}$$

The concurrence of the spin squeezed states is given by analytical expressions in the argument $\mu = 2\chi t$, which are too lengthy to present here. In Figure 2 we present the results numerically: if two atoms are extracted at random from spin squeezed samples they will be in a mutually entangled state. We observe that the concurrence is symmetric with respect to $\mu = \pi$. At this special point of $\mu = \pi$ the N-particle GHZ state is produced [6], which has no entanglement in any subset of particles, hence the concurrence vanishes at this point.

5 Mixed multiqubit states and thermal entanglement

An interesting and novel type of thermal entanglement was introduced and analyzed within the Heisenberg XXX [19], XX [20], and XXZ [21] models as well as within the Ising model in a magnetic field [22]. The state of the system at thermal equilibrium is represented by the density operator $\rho(T) = \exp(-H/kT)/Z$, where $Z = \mathrm{tr}[\exp(-H/kT)]$ is the partition function, H the system Hamiltonian, k is Boltzmann's constant which we henceforth take equal to unity, and T the temperature. As $\rho(T)$ represents a thermal state, the entanglement in the state is called *thermal entanglement* [19]. Unlike in standard statistical physics where all properties are obtained from the partition function, determined by the eigenvalues of the system, entanglement properties require in addition knowledge of the eigenstates. The analytical results in the previous studies on thermal entanglement are only available for two [19–22] and three qubits [23]. Here we consider pairwise entanglement in the multiqubit systems.

5.1 Isotropic Heisenberg model

We consider the N-qubit isotropic Heisenberg Hamiltonian

$$H_I = \frac{J}{4}\sum_{i\neq j}^{N}\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z\right). \tag{22}$$

The positive (negative) J corresponds to the antiferromagnetic (ferromagnetic) case. In this model all particles interact with each other.

By using the collective spin operators, the Hamiltonian is rewritten as

$$H = J\left(S_x^2 + S_y^2 + S_z^2\right) = J\mathbf{S}^2 \tag{23}$$

up to a trivial constant.

Unlike pure states, the symmetric multi-particle density matrix does not only populate the fully symmetric Dicke states, and we have to determine the number of collective spin-S states for each S. Write S as $(N/2-k)$, we know that for $k = 0$, a single irreducible representation exists: the $N+1$ fully symmetric Dicke states with $S = N/2$. The number of irreducible representation with $S = N/2 - 1$ is obtained by noting that their maximum M value is also $N/2-1$, and a total of $\binom{N}{1} = N$ states exist with precisely one particle in the $|1\rangle$ state. One of these belong to the $S = N/2$ irreducible representation, and the remaining $N-1$ states must have $S = N/2-1$. This argument can now be repeated to obtain the number of states with $S = N/2-2$ and $M = N/2-2$, *i.e.*, the number of $S = N/2-2$ irreducible representation, etc., until all 2^N states of the system have been accounted for.

The isotropic Hamiltonian only depends on $\mathbf{S}^2$, and knowing the multiplicity of each value of this quantity we write the partition function

$$Z = \sum_{k=0}^{N/2} N_k [2(N/2-k)+1] \mathrm{e}^{-\beta J(N/2-k)(N/2-k+1)}, \quad (24)$$

where $N_k = \binom{N}{k} - \binom{N}{k-1}$ follows from the above argument. We assume $\binom{N}{-1} = 0$.

The reduced density matrix for two qubits is given by

$$\rho_{12} = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & y & 0 \\ 0 & y & w & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \quad (25)$$

with matrix elements

$$\begin{aligned} y &= \frac{2\langle S_x^2 + S_y^2 \rangle - N}{2N(N-1)}, \\ w &= \frac{N^2 - 4\langle S_z^2 \rangle}{4N(N-1)}, \\ v &= \frac{N^2 - 2N + 4\langle S_z^2 \rangle}{4N(N-1)}, \end{aligned} \quad (26)$$

where

$$\langle S_z^2 \rangle = \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} (m - N/2 + k)^2 \times \mathrm{e}^{-\beta J(N/2-k)(N/2-k+1)}/Z. \quad (27)$$

Note that $y \neq w$ in this case.

From equations (16, 26), the concurrence is obtained as

$$\begin{aligned} \mathcal{C} &= \frac{1}{2N(N-1)} \\ &\quad \times \max\left\{0, 2|2\langle S_x^2 + S_y^2\rangle - N| - N^2 + 2N - 4\langle S_z^2\rangle\right\} \\ &= \frac{1}{2N(N-1)} \\ &\quad \times \max\left\{0, 2|4\langle S_z\rangle - N| - N^2 + 2N - 4\langle S_z^2\rangle\right\}, \end{aligned} \quad (28)$$

where we have used the symmetric property $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle$.

To identify the sign of $A \equiv 2|4\langle S_z^2\rangle - N| - N^2 + 2N - 4\langle S_z^2\rangle$ in (28), we consider the case where $4\langle S_z^2\rangle \geq N$, for which $A = 4\langle S_z^2\rangle - N^2$. Since

$$\langle S_z^2 \rangle \leq \frac{1}{3}\frac{N}{2}\left(\frac{N}{2}+1\right) < \frac{N^2}{4},$$

we always have $4\langle S_z^2\rangle - N^2 < 0$, and there is no pairwise entanglement. In the opposite case where $4\langle S_z^2\rangle < N$, we have $A = 4N - 12\langle S_z^2\rangle - N^2$. For $N \geq 4$ this quantity is always negative, and since for any spin S, $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq S$, we know that for $N = 3$, we have $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq 3/2$. Due to symmetry we thus have $\langle S_z^2\rangle \geq 1/2$, and A is always negative if $N \geq 3$.

So we conclude that there is no thermal entanglement for $N \geq 3$ in the isotropic Heisenberg model. The case of $N = 2$ is discussed in detail in reference [19] and it is shown that there is no thermal entanglement for the ferromagnetic case. In order to observe pairwise entanglement in the multiqubit system, we now consider the anisotropic Heisenberg model.

5.2 Anisotropic Heisenberg model

The anisotropic Heisenberg Hamiltonian is given by

$$H_a = J\left(S_x^2 + S_y^2 + \Delta S_z^2\right) = J\mathbf{S}^2 + J(\Delta - 1)S_z^2, \quad (29)$$

where Δ is the anisotropic parameter. Obviously the Hamiltonian H_a reduces to H_I when $\Delta = 1$, and H_a yields the XX model when $\Delta = 0$.

The concurrence is still given by (28), but the partition function and the relevant expectation values now become

$$\begin{aligned} Z &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} \mathrm{e}^{-\beta J(\Delta-1)(m-N/2+k)^2} \\ &\quad \times \mathrm{e}^{-\beta J(N/2-k)(N/2-k+1)}, \end{aligned} \quad (30)$$

$$\begin{aligned} \langle S_z^2 \rangle &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} (m-N/2+k)^2 \\ &\quad \times \mathrm{e}^{-\beta J(\Delta-1)(m-N/2+k)^2} \\ &\quad \times \mathrm{e}^{-\beta J(N/2-k)(N/2-k+1)}/Z, \end{aligned} \quad (31)$$

$$\begin{aligned} \langle S_x^2 + S_y^2 \rangle &= \sum_{k=0}^{N/2} N_k \sum_{m=0}^{N-2k} [(N/2-k)(N/2-k+1) \\ &\quad -(m-N/2+k)^2] \\ &\quad \times \mathrm{e}^{-\beta J(\Delta-1)(m-N/2+k)^2} \\ &\quad \times \mathrm{e}^{-\beta J(N/2-k)(N/2-k+1)}/Z. \end{aligned} \quad (32)$$

This model leads to pairwise entanglement, as shown by the numerical results presented in Figure 3 as functions of the reciprocal temperature, $x = \beta J$. For $N = 2$ we observe that the concurrence is symmetric with respect to $x = 0$, which is consistent with the result in reference [20]. In other words, the thermal entanglement appears for both the antiferromagnetic and ferromagnetic cases. However for $N \geq 3$, the thermal entanglement only exists for the ferromagnetic case. We observe a critical value of x, after which the entanglement vanishes. And the critical value increases as N increases.

Within the above framework we may consider more general models such as

$$H_g = J\mathbf{S}^2 + f(S_z), \quad (33)$$

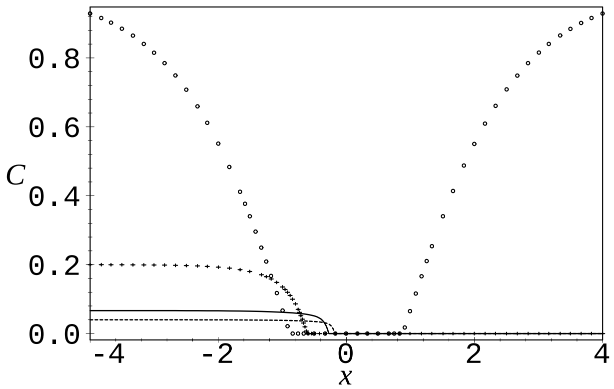

Fig. 3. The concurrence as a function of $x = \beta J$ for different number N in the XX model ($\Delta = 0$): $N = 2$ (open circle), $N = 5$ (crosses), $N = 15$ (solid line), and $N = 25$ (dashed line).

where $f(S_z)$ is an arbitrary analytical function of S_z. As the operator $f(S_z)$ commutes with $\mathbf{S}^2$, similar analytical results for the concurrence can be obtained and the thermal entanglement can be generated for special choices of $f(S_z)$.

6 EPR-correlated ensembles

Finally we consider two EPR-correlated ensembles. This state is not invariant under any permutation of particles, but only under these permutations that exchange particles within each ensemble, and it is furthermore characterized by the correlations between the samples 1 and 2:

$$(J_{1x} - J_{2x})|\Psi\rangle = 0, \quad (34)$$

$$(J_{1y} + J_{2y})|\Psi\rangle = 0. \quad (35)$$

A state that obeys equations (34, 35) can in principle be obtained by QND detection of the observables $J_{1x} - J_{2x}$ and $J_{1y} + J_{2y}$ [10,11]. Equivalently the above equations can be written as

$$(J_{1+} - J_{2-})|\Psi\rangle = 0, \quad (36)$$

$$(J_{1-} - J_{2+})|\Psi\rangle = 0. \quad (37)$$

It is easy to check that a solution of the above equation is the EPR-correlated state

$$|\Psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n\rangle_N \otimes |n\rangle_N \quad (38)$$

and it also satisfies $(J_{1z} - J_{2z})|\Psi\rangle = 0$. The entanglement of formation of $|\Psi\rangle$ is easily obtained as $E = \log_2(N+1)$.

Now we consider the entanglement of two qubits, which belong to different ensembles. we first identify the two-qubit reduced density matrix:

$$\rho_{12} = \begin{pmatrix} v_+ & 0 & 0 & u^* \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ u & 0 & 0 & v_- \end{pmatrix} \quad (39)$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which can be represented by

$$\begin{aligned} v_{\pm} &= \frac{1}{4}\left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right), \\ w &= \frac{1}{4}\left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right) = \frac{1}{4} - \frac{\langle J_{1z}J_{2z}\rangle}{N^2}, \\ u &= \frac{1}{4}(\langle\sigma_{1x}\sigma_{2x}\rangle - \langle\sigma_{1y}\sigma_{2y}\rangle + \mathrm{i}2\langle\sigma_{1x}\sigma_{2y}\rangle) \\ &= \frac{\langle J_{1+}J_{2+}\rangle}{N^2}. \end{aligned} \quad (40)$$

The concurrence is given by

$$\begin{aligned} \mathcal{C} &= 2\max\{0, |u| - w\} \\ &= 2\max\left\{0, \frac{\langle J_{1+}J_{2+}\rangle + \langle J_{1z}J_{2z}\rangle}{N^2} - \frac{1}{4}\right\}. \end{aligned} \quad (41)$$

One simple observation is that the concurrence is independent on $\langle\sigma_{1z}\rangle$. The expectation values of $J_{1+}J_{2+}$ and $J_{1z}J_{2z}$ are obtained as

$$\begin{aligned} \langle J_{1+}J_{2+}\rangle &= \frac{1}{N+1}\sum_{n=0}^{N-1}(n+1)(N-n) \\ \langle J_{1z}J_{2z}\rangle &= \frac{1}{N+1}\sum_{n=0}^{N}(n-N/2)^2. \end{aligned} \quad (42)$$

Finally we find that $\mathcal{C} = 1/N$.

7 Conclusions

The purpose of this paper has been to investigate to which extent multi-particle entanglement implies pairwise entanglement within the sample. We showed that the two-particle density matrix is readily expressed in terms of expectation values of collective operators in the case of symmetrical states of the many-particle system, and we provided the value of the concurrence for a number of examples. These results confirmed and generalized results obtained, *e.g.*, on the pairwise entanglement in systems with definite ($N = 3, 4$) numbers of particles. Most of our examples of multi-particle entangled states showed some degree of entanglement of the extracted pair of particles. From a sample of $N/2$ pairs, it is thus possible to distill a single pair with a higher degree of entanglement [13,24,30]. Note, however, that substantial entanglement is lost when the multi-particle state is partitioned in pairs, and a much better protocol may be envisioned for the production of a an entangled pair of qubits from the multi-particle entangled state.

The entanglement of formation and the very issue of entanglement are highly non-trivial for situations dealing with more than two particles, and for mixed states of systems with dimensions higher than 2. Studying and optimizing the two-particle concurrence in systems with many particles may be a useful way to learn about the more complicated case. A next step could be, for example, to apply the method of Section 3, to obtain the state of three particles, where different non-equivalent kinds of entanglement may be identified.

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References

1. W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A **62**, 062314 (2000); W. Dür, Phys. Rev. A **63**, 020303 (2001).
2. K. Mølmer, Eur. Phys. J. D **5**, 301 (1998).
3. A. Kuzmich, N.P. Bigelow, L. Mandel, Europhys. Lett. **42**, 481 (1998).
4. A. Kuzmich, L. Mandel, N.P. Bigelow, Phys. Rev. Lett. **85**, 1594 (2000).
5. M. Kitagawa, M. Ueda, Phys. Rev. A **47**, 5138 (1993).
6. K. Mølmer, A. Sørensen, Phys. Rev. Lett. **82**, 1835 (1999).
7. C.A. Sackett *et al.*, Nature **404**, 256 (2000).
8. A. Sørensen, L.-M. Duan, I. Cirac, P. Zoller, Nature **409**, 63 (2001).
9. U. Poulsen, K. Mølmer, Phys. Rev. A **64**, 013616 (2001).
10. L.-M. Duan, J.I. Cirac, P. Zoller, E.S. Polzik, Phys. Rev. Lett. **85**, 5643 (2000).
11. B. Julsgaard, A. Kozhekin, E.S. Polzik, Nature **413**, 400 (2001).
12. G. Vidal, W. Dür, J.I. Cirac, Phys. Rev. Lett. **85**, 658 (2000); W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A **62**, 062314 (2000).
13. W. Dür, J.I. Cirac, Phys. Rev. A **61**, 042314 (2000); W. Dür, R. Tarrach, J.I. Cirac, Phys. Rev. Lett. **83**, 3562 (1999).
14. C.H. Bennett *et al.*, Phys. Rev. A **63**, 012307 (2001).
15. J. Eisert, H.-J. Briegel, Phys. Rev. A **64**, 022306 (2001).
16. S. Hill, W.K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
17. W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
18. C.H. Bennett, D.P. DiVincenzo, Nature **404**, 247 (2000).
19. M.C. Arnesen, S. Bose, V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001); M.A. Nielsen, Ph.D. thesis, University of Mexico, 1998, `quant-ph/0011036`.
20. X. Wang, Phys. Rev. A **64**, 012313 (2001).
21. X. Wang, Phys. Lett. A **281**, 101 (2001).
22. D. Gunlycke, S. Bose, V.M. Kendon, V. Vedral, `quant-ph/0102137`.
23. X. Wang, H. Fu, A.I. Solomon, `quant-ph/0105075`.
24. C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
25. A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A **232**, 333 (1997).
26. R.H. Dicke, Phys. Rev. **93**, 99 (1954).
27. J.M. Radcliffe, J. Phys. A: Gen. Phys. **4**, 313 (1971); F.T. Arecchi, E. Courtens, R. Gilmore, H. Thomas, Phys. Rev. A **6**, 2211 (1972).
28. K.M. O'Connor, W.K. Wootters, Phys. Rev. A **63**, 052302 (2001).
29. M. Koashi, V. Bužek, N. Imoto, Phys. Rev. A **62**, 050302 (2000).
30. C.H. Bennett *et al.*, Phys. Rev. Lett. **76**, 722 (1996).